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On resolving the multiplicity of the branching rule $GL(2k, \mathbb{C})\downarrow Sp(2k, \mathbb{C})$

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Abstract. We consider the multiplicity problem of the branching rule $GL(2k, \mathbb{C})\downarrow Sp(2k, \mathbb{C})$. Finite-dimensional irreducible representations of $GL(2k, \mathbb{C})$ are realized as right translations on subspaces of the holomorphic Hilbert (Bargmann) spaces of $q\times 2k$ complex variables. Maps are exhibited which carry an irreducible representation of $Sp(2k, \mathbb{C})$ into these subspaces. An algebra of commuting operators is constructed and it is shown how eigenvalues of certain of these operators can be used to resolve the multiplicity.

1. Introduction

One of the outstanding problems in the representation theory of Lie groups is the branching rule problem. Let G be a given Lie group and H a subgroup of G. Then it is well known that a finite-dimensional irreducible representation of G can be decomposed as a direct sum of irreducible representations of H. The same irreducible representation of H may appear more than once in the decomposition. The branching rule $G\downarrow H$ consists of finding the multiplicity of an irreducible representation of H that occurs in the decomposition. Whippman [14] stated several branching rules associated with simple Lie groups. Since then, different techniques and formulations have been developed for studying branching rules of classical Lie groups [3, 5, 11, 16] and of certain exceptional groups [8, 15]. In this paper, we want to study the branching rule $GL(2k, \mathbb{C})\downarrow Sp(2k, \mathbb{C})$. In [5, 11], the rules are formulated via the Young tableaux. King [3] formulated the rule in terms of the addition and division of certain symmetric functions known as S-functions. These branching rules only give the multiplicity of an irreducible representation of H and do not distinguish the equivalent irreducible representations. Here, our emphasis is different. We want to find a canonical way of labelling the equivalent representations that occur in the branching rule $GL(2k, \mathbb{C})\downarrow Sp(2k, \mathbb{C})$, in other words, we want to break the multiplicity that appears in the branching rule explicitly. We shall construct maps that carry an irreducible representation of $Sp(2k, \mathbb{C})$ into an irreducible representation of $GL(2k, \mathbb{C})$. Then a class of generalized commuting Casimir operators are exhibited. The eigenvalues and eigenvectors of these operators can then be used to break the multiplicity that occurs in the branching rule. The main tools needed to carry out this analysis are a Fock space in $q\times 2k$ complex variables and the theory of dual pairs [2, 9], which is used to construct the Casimir operators.

The general set-up for our problem will be discussed in section 2. Section 3 makes use of the notion of dual pairs to exhibit an algebra of commuting differential operators.

Theorem 3.3 gives explicit form of this algebra of operators; the eigenvalues and eigenvectors of certain elements of this algebra can then be used as labels to distinguish the equivalent representations that occur in the branching rule. This is shown explicitly in section 4 with an example of the branching rule $GL(8, \mathbb{C})\downarrow Sp(8, \mathbb{C})$.

2. The general set-up

From now on, we let G denote the general linear group $GL(2k, \mathbb{C})$ and H denote the complex symplectic group $Sp(2k, \mathbb{C})$. It is clear that H is embedded in G in a natural way. The irreducible representation of G are parametrized by a $2k$-tuple of non-negative integers $(m)=(m_1, \ldots, m_{2k})$ which satisfies the dominant condition $m_1 \geqslant \ldots \geqslant m_{2k}$. A concrete realization of a finite-dimensional irreducible representation of G can be obtained in the following fashion. Suppose $(m)=(m_1, \ldots, m_{2k})$ such that $m_{q+1}= \ldots = m_{2k}=0$ for some $1 \leqslant q \leqslant 2k$. If B_q denotes the subgroup of lower triangular matrices of G, then we define a holomorphic character

$$\pi^{(m)}: B_q \rightarrow \mathbb{C}^*$$

$$\pi^{(m)}(b)=b_{11}^{m_1} \ldots b_{qq}^{m_q} \qquad \forall b \in B_q .$$

Let $\mathbb{C}^{q \times 2k}$ denote the complex vector space of all $q \times 2k$ matrices. Let $V_{GL}^{(m)}$ denote the complex vector space of all polynomial functions $f: \mathbb{C}^{q \times 2k} \rightarrow \mathbb{C}$ which satisfy the covariant condition $f(bZ)=\pi^{(m)}(b)f(Z)$, for all (b, Z) belonging to $B_q \times \mathbb{C}^{q \times 2k}$. Let $R_{GL}^{(m)}$ denote the holomorphically induced representation of G on $V_{GL}^{(m)}$ defined by $(R_{GL}^{(m)}(g)f)(Z)=f(Zg)$, $g \in G$. Then according to [12], $R_{GL}^{(m)}$ is irreducible and its highest weight is indexed by (m) which is called the *signature of the representation* $R_{GL}^{(m)}$. However, if we restrict this representation to H, then $R_{GL}^{(m)}$ becomes a reducible representation of H.

Now, if $Z=[z_{\alpha\beta}] \in \mathbb{C}^{q \times 2k}$, then set $\bar{Z}$ to be the complex conjugate of Z. Let $\mathrm{d}X_{\alpha i}$ and $\mathrm{d}Y_{\alpha i}$ denote the Lebesgue measure on $\mathfrak{R}$ and define

$$\mathrm{d}Z=\prod_{\alpha=1}^{q} \prod_{i=1}^{2k} \mathrm{d}X_{\alpha i}\, \mathrm{d}Y_{\alpha i}$$

to be the Lebesgue product measure on $\mathfrak{R}^{2qk}$. Define a Gaussian measure

$$\mathrm{d}\mu(Z)=\pi^{-2qk} \exp[-\mathrm{tr}(Z\bar{Z}')]$$

$Z \in \mathbb{C}^{q \times 2k}$, where tr denote the trace of a matrix. A map $f: \mathbb{C}^{q \times 2k} \rightarrow \mathbb{C}$ is called holomorphic square integrable if it is holomorphic on the entire domain $\mathbb{C}^{q \times 2k}$ and if

$$\int_{\mathbb{C}^{q \times 2k}} |f(Z)|^2\, \mathrm{d}\mu(Z) < \infty .$$

It is obvious that the holomorphic square integrable functions form a complex vector space, in fact, they form a Hilbert space with respect to the inner product

$$(f_1, f_2)=\int_{\mathbb{C}^{q \times 2k}} f_1(Z)\overline{f_2(Z)}\, \mathrm{d}\mu(Z).$$

Let $\mathscr{F} \equiv \mathscr{F}(\mathbb{C}^{q \times 2k})$ denote this Hilbert space, which is known as the Fock space of $q \times 2k$ complex variables. If $P(\mathbb{C}^{q \times 2k})$ is the vector space of all polynomial functions on $\mathbb{C}^{q\times 2k}$, then $P(\mathbb{C}^{q\times 2k})$ is a dense subspace in $\mathscr{F}$. Moreover, if we endow $P(\mathbb{C}^{q\times 2k})$ with the differential inner product

$$\langle p_1, p_2\rangle = p_1(D)\overline{p_2(\bar{Z})}|_{z=0} \tag{2.1}$$

where $p(D)$ denotes the differential operator by replacing z_{ij} by $\partial/\partial z_{ij}$ ($1\leqslant i\leqslant q$, $1\leqslant j\leqslant 2k$), then we can easily verify that the inner product $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are identical on $P(\mathbb{C}^{q\times 2k})$.

If D_q denote the group of all complex diagonal invertible matrices of order q, and if $(M)=(M_1,\ldots,M_q)$ is a q-tuple of non-negative integers, we define a holomorphic character

$$\zeta^{(M)}: D_q\to\mathbb{C}^*$$

$$\zeta^{(M)}(d)=d_{11}^{M_1}\ldots d_{qq}^{M_q} \qquad \forall d\in D_q.$$

A polynomial function $p: \mathbb{C}^{q\times 2k}\to\mathbb{C}$ is said to transform covariantly with respect to $\zeta^{(M)}$ if $f(\mathrm{d}Z)=\zeta^{(M)}(d)f(Z)$, for all (d, Z) belonging to $D_q\times\mathbb{C}^{q\times 2k}$. It is obvious that the polynomial functions which transform covariantly with respect to $\zeta^{(M)}$ form a subspace of $\mathscr{F}$. We shall denote this subspace by $P^{(M)}$. Now, if $(m)=(m_1,\ldots,m_q)$ is a q-tuple of integers such that $m_1\geqslant\ldots\geqslant m_q\geqslant 0$, then it is clear that $V_{GL}^{(m,0,\ldots,0)}$ is a subspace of $P^{(m)}$.

Now, an irreducible holomorphically induced representation of H is parametrized by a k-tuple of non-negative integers $(m)=(m_1,\ldots,m_k)$ such that $m_1\geqslant\ldots\geqslant m_k$. A concrete realization of such an irreducible representation of H can be obtained as follows. Let B_k denote the lower triangular subgroup of $GL(k,\mathbb{C})$. We define a holomorphic character

$$\xi^{(m)}: B_k\to\mathbb{C}^*$$

$$\xi^{(m)}(b)=b_{11}^{m_1}\ldots b_{kk}^{m_k} \qquad \forall b\in B_k.$$

Consider the following space:

$$V_{Sp}^{(m)}=\Bigg\{f:\mathbb{C}^{k\times 2k}\to\mathbb{C}\,|\, f \text{ polynomial function}, f(bX)=\xi^{(m)}(b)f(X),$$

$$\text{for } b\in B_q, X\in\mathbb{C}^{k\times 2k} \text{ and } \sum_{p=1}^{k}\left(\frac{\partial^2 f}{\partial Z_{ip}\,\partial Z_{jp+k}}-\frac{\partial^2 f}{\partial Z_{ip+k}\,\partial Z_{jp}}\right)=0,\ 1\leqslant i<j\leqslant k\Bigg\}.$$

Let $R_{Sp}^{(m)}$ denote the representation of H on $V_{Sp}^{(m)}$ by right translation, that is, $(R_{Sp}^{(m)}(h)f)(Z)=f(Zh)$, $h\in H$. Then according to [13], $R_{Sp}^{(m)}$ is irreducible with signature (m).

Finally, we conclude this section with the following definition.

Definition 2.2. Let $V_{Sp}^{(m)}$ be an H-module. The isotypic component $I(V_{Sp}^{(m)})$ of $V_{Sp}^{(m)}$ in $\mathscr{F}$ is the sum of all H-modules in $\mathscr{F}$ which are equivalent to $V_{Sp}^{(m)}$.

3. The multiplicity breaking of $G\downarrow H$

We shall now give a procedure for breaking the multiplicity that appears in the branching rule $G\downarrow H$. For this, suppose $(m)=(m_1,\ldots,m_q)$ is a q-tuple of integers such that

$m_1 \geqslant \ldots \geqslant m_q \geqslant 0$. Let $L^{(m)}$ denote the representation of $GL(q, \mathbb{C})$ on $P^{(m)}$ defined by $(L^{(m)}(g)p)(Z) = p(g^{-1}Z)$, $g \in GL(q, \mathbb{C})$ and $R^{(m)}$ denote the representation of G on $P^{(m)}$ by right translation. If we let L_{ij} (respectively, R_{rs}) denote the infinitesimal operators of $L^{(m)}$ (respectively, $R^{(m)}$) corresponding to the standard basis e_{ji} (respectively, e_{rs}) of the Lie algebra $\mathbb{C}^{q \times q}$ (respectively, $\mathbb{C}^{2k \times 2k}$) of $GL(q, \mathbb{C})$ (respectively, G); then

$$L_{ij} = \sum_{\eta=1}^{2k} Z_{i\eta} \frac{\partial}{\partial Z_{j\eta}} \qquad R_{rs} = \sum_{\eta=1}^{q} Z_{\eta r} \frac{\partial}{\partial Z_{\eta s}} \qquad 1 \leqslant i, j \leqslant q \qquad 1 \leqslant r, s \leqslant 2k.$$

Now, it is easy to see that the space $V_{GL}^{(m)}$ consists of polynomial functions in $P^{(m)}$ which are simultaneously annihilated by all lowering operators of the form

$$L_{ij} \qquad \text{with } 1 \leqslant i < j \leqslant q. \tag{3.1}$$

and let us denote by $(R_{GL}^{(m)}|_H, V_{GL}^{(m)})$ the restriction of the representation of $R_{GL}^{(m)}$ to H.

Now, let $SU(q, q)$ be the linear isometry group for the Hermitian form

$$|Z_1|^2 + |Z_2|^2 + \ldots + |Z_q|^2 - |Z_{q+1}|^2 - \ldots - |Z_{2q}|^2 \qquad \text{over } \mathbb{C}.$$

Then the group $SO^*(2q)$ is the set of all elements g in $SU(q, q)$ such that $g^{\mathrm{T}} J g = J$, where g^{T} is the transpose of g and

$$J = \begin{pmatrix} 0 & I_q \\ I_q & 0 \end{pmatrix}$$

where I_q is the $q \times q$ identity matrix. Then according to [2, 7], $(SO^*(2q), Sp(2k, \mathbb{C}))$ forms a dual pair of reductive groups. Let

$$L_{ij} = \sum_{\eta=1}^{2k} Z_{i\eta} \frac{\partial}{\partial Z_{j\eta}} \qquad P_{ij} = \sum_{\eta=1}^{k} (Z_{i\eta+k} Z_{j\eta} - Z_{i\eta} Z_{j\eta+k})$$

and

$$D_{ij} = \sum_{\eta=1}^{k} \left(\frac{\partial^2}{\partial Z_{i\eta+k}\, \partial Z_{j\eta}} - \frac{\partial^2}{\partial Z_{i\eta}\, \partial Z_{j\eta+k}} \right) \qquad 1 \leqslant i, j \leqslant q. \tag{3.2}$$

These operators form a basis for the Lie algebra $so^*(2q)$ of the group $SO^*(2q)$ and they generate a universal enveloping algebra $\mathscr{U}$ of differential operators which acts on $P(\mathbb{C}^{q \times 2k})$. Moreover, by the Poincaré–Birkoff–Witt theorem, the ordered monomials in L_{ij}, P_{ij}, D_{ij} form a basis for the algebra $\mathscr{U}$.

Now suppose the H-module $(R_{Sp}^{(m')}, V_{Sp}^{(m')})$ occurs in $P^{(m)}$ μ times (as will be shown in section 4, we can compute this μ easily with the help of a special formula). Then from a consequence of Burnside's theorem and the theory of dual pairs [2, 9], there exist μ linearly independent elements in $\mathscr{U}$ which form a basis for the vector space $\mathrm{Hom}_H(V_{Sp}^{(m')}, P^{(m)})$ of all intertwining operators from $V_{Sp}^{(m')}$ to $P^{(m)}$. In fact, if $h_{\max}^{(m')}$ is the highest weight vector of $V_{Sp}^{(m')}$, then one can choose μ elements $p_1, \ldots, p_\mu$ of $\mathscr{U}$ such that $p_i h_{\max}^{(m')}$, $1 \leqslant i \leqslant \mu$, are linearly independent highest-weight vectors of the μ copies of the H-module equivalent to $V_{Sp}^{(m')}$ which are contained in $P^{(m)}$. Let $W_{\max}^{(m')(m)}$ denote the vector space spanned by $p_i h_{\max}^{(m')}$, and let $\mathrm{Ker}_{\max}^{(m')(m)}$ denote its projection in $V_{GL}^{(m)}$, that is, $\mathrm{Ker}_{\max}^{(m')(m)}$ is the common kernel subspace in $W_{\max}^{(m')(m)}$ of all operators L_{ij} in (3.1). Hence, $\mathrm{Ker}_{\max}^{(m')(m)}$ is isomorphic to the subspace $I(V_{Sp}^{(m')}) \cap (R_{GL}^{(m)}|_H, V_{GL}^{(m)})$, the intersection of the isotypic component of $V_{Sp}^{(m')}$ with $(R_{GL}^{(m)}|_H, V_{GL}^{(m)})$. In order to break the multiplicity in $I(V_{Sp}^{(m')}) \cap (R_{GL}^{(m)}|_H, V_{GL}^{(m)})$, we shall find operators in $\mathscr{U}$ which commute with the operators L_{ij} in (3.1) (but without the condition $i<j$) and which decompose $\mathrm{Ker}_{\max}^{(m')(m)}$ into distinct one-dimensional eigenspaces. Then the eigenvalues and eigenvectors associated with the operators can be used as labels to distinguish the equivalent representations. In fact, we shall use commuting Hermitian operators in $\mathscr{U}$ so that the eigenvectors are all orthogonal to each other with respect to the inner product (2.1).

Let us now concentrate on finding operators in $\mathscr{U}$ which commute with L_{ij}. Let Z be an element of $\mathbb{C}^{q\times 2k}$. The action of $GL(q, \mathbb{C})$ on Z is simply of the form

$$g'\to[g'Z] \qquad g'\in GL(q, \mathbb{C}).$$

Its dual action is therefore

$$g\to[Zg] \qquad g\in G.$$

Now, let R denote the matrix (R_{rs}). We write L_{ij}, $1\leqslant i, j\leqslant q$, into a $q\times q$ matrix $[L]$, that is,

$$[L]=\begin{bmatrix} L_{11} & \cdots & L_{1q} \\ \vdots & \ddots & \vdots \\ L_{q1} & \cdots & L_{qq} \end{bmatrix}.$$

Similarly, we write P_{ij} (respectively, D_{ij}), $1\leqslant i, j\leqslant q$, into a $q\times q$ matrix $[P]$ (respectively, $[D]$). Also, let $[E]$ denote the matrix $[-L]^{\mathrm{T}}$, where T denotes the transpose. We now have the main theorem of this paper.

Theorem 3.3. In the universal enveloping algebra $\mathscr{U}$, consider the trace of arbitrary products of the following matrices:

(1) $[L]$

(2) $[P]\,[D]$ (3.4)

(3) $[P]\,[E]\,[D]$.

Then these operators generate a subalgebra $\mathscr{V}$ of differential operators in $\mathscr{U}$ that commute with the operators L_{ij}, $l\leqslant i, j\leqslant q$.

Example. We can form the following commuting operator:

$$\mathrm{tr}([P][D][L][P][E][D]).$$

Proof. Let Γ' denote the complexification of $so^*(2q)$, that is $\Gamma'=so(2q, \mathbb{C})$, the Lie algebra of the complex $2q\times 2q$ special orthogonal group. If $\xi\in\Gamma'$, then ξ is of the following form:

$$\xi=\begin{pmatrix} Y & X \\ W & Q \end{pmatrix}$$

where $[Y]$ is a $q\times q$ complex matrix, $[Q]=-[Y]^{\mathrm{T}}$ and $[W]$, $[X]$ are $q\times q$ skew-symmetric matrices. On the other hand, the differential operators P_{ij}, D_{ij} and L_{ij} as defined in (3.2) also form a basis for Γ'. Let $S(so(2q, \mathbb{C}))$ denote the symmetric algebra of $so(2q, \mathbb{C})$, and $\mathscr{U}(so(2q, \mathbb{C}))$ denote the universal enveloping algebra of $so(2q, \mathbb{C})$. We can now define the co-adjoint representation T of $H' = SO(2q, \mathbb{C})$ in $S(so(2q, \mathbb{C}))$ by the equation

$$[T(h')p](\xi) = p(h'^{-1}\xi h') \qquad h' \in H', p \in S(so(2q, \mathbb{C})) \text{ and } \xi \in \Gamma'.$$

A polynomial $p \in S(so(2q, \mathbb{C}))$ is said to be K-invariant, where K is a subgroup of H', if $T(k)p = p$, for all $k \in K$.

We now have the canonical isomorphism ϕ of $S(so(2q, \mathbb{C}))$ on to $\mathscr{U}(so(2q, \mathbb{C}))$ defined as follows (cf [1]).

Suppose $p \in S(so(2q, \mathbb{C}))$, then p can be expressed uniquely as

$$p(\xi) = \sum_{s \leqslant d} a_{i_1 j_1 \ldots i_s j_s} y_{i_1 j_1} \ldots x_{i_p j_p} \ldots w_{i_s j_s}$$

where the coefficients $a_{i_1 j_1 \ldots i_s j_s}$ are symmetric functions, that is, $a_{i_{\sigma(1)} j_{\sigma(1)} \ldots i_{\sigma(s)} j_{\sigma(s)}} = a_{i_1 j_1 \ldots i_s j_s}$, for all permutations σ in the symmetric group of order s and for all integers s less than or equal to a fixed integer d. Now, $\phi: S(so(2q, \mathbb{C})) \to \mathscr{U}(so(2q, \mathbb{C}))$ is defined by

$$\phi(p) = \sum_{s \leqslant d} a_{i_1 j_1 \ldots i_s j_s} L_{i_1 j_1 \ldots} P_{i_p j_p \ldots} D_{i_s j_s}.$$

Let K be the a diagonal subgroup of $SO(2q, \mathbb{C})$ defined as follows:

$$K = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & \tilde{k}_1 \end{pmatrix} : k_1 \in GL(q, \mathbb{C}) \right\}.$$

An element $u \in \mathscr{U}(so(2q, \mathbb{C}))$ is said to be K-invariant if the conditions

$$[u, L_{ij}] = 0 \qquad \forall L_{ij},\ 1 \leqslant i, j \leqslant q$$

are satisfied. Observe that L_{ij} generate the Lie algebra of K.

It is well known that the map ϕ carries the K-invariant polynomials onto the K-invariant differential operators (cf [1]). Thus, to show that a differential operator of the form (3.4) is K-invariant, it suffices to show that its inverse image under the map ϕ is a K-invariant polynomial function. For this, we let

$$k = \begin{pmatrix} k_1 & \\ & \tilde{k}_1 \end{pmatrix}$$

be an element of K and let $\xi \in \Gamma'$. The matrix $k^{-1}\xi k$, $k \in K$, can be written as

$$k^{-1}\xi k = \begin{pmatrix} k_1^{-1}[Y]k_1 & k_1^{-1}[X]\tilde{k}_1 \\ \tilde{k}_1^{-1}[W]k_1 & \tilde{k}_1^{-1}[Q]\tilde{k}_1 \end{pmatrix}.$$

If f_1 is the inverse image under the map ϕ of the type 1 in (3.4), then f_1 is given by

$$f_1(\xi) = \operatorname{tr}([Y])$$

and

$$\begin{aligned}[T(k)f_1](\xi) &= f_1(k^{-1}\xi k) \\ &= \operatorname{tr}(k_1^{-1}[Y]k_1) \\ &= f_1(\xi).\end{aligned}$$

If f_2 is the inverse image of ϕ type 2 in (3.4), then f_2 is given by

$$f_2(\xi) = \operatorname{tr}([X][W])$$

and

$$\begin{aligned}[T(k)f_2](\xi) &= f_2(k^{-1}\xi k) \\ &= \mathrm{tr}((k_1^{-1}[X]\tilde{k}_1)(\tilde{k}_1^{-1}[W]_1 k_1)) \\ &= \mathrm{tr}(k_1^{-1}[X][W]k_1) \\ &= \mathrm{tr}([X][W]) \\ &= f_2(\xi).\end{aligned}$$

If f_3 is the inverse image of ϕ of type 3 in (3.4), then f_3 is given by

$$f_3(\xi) = \mathrm{tr}([X][Q][W])$$

and

$$\begin{aligned}[T(k)f_3](\xi) &= f_3(k^{-1}\xi k) \\ &= \mathrm{tr}((k_1^{-1}[X]\tilde{k}_1)(\tilde{k}_1^{-1}[Q]\tilde{k}_1)(\tilde{k}_1^{-1}[W]k_1)) \\ &= \mathrm{tr}(k_1^{-1}[X][Q][W]k_1) \\ &= \mathrm{tr}([X][Q][W]) \\ &= f_3(\xi).\end{aligned}$$

Now, it is clear that besides the commuting operators obtained from types 1, 2 and 3, we can multiply them together to form new commuting operators. Up to now, we have been dealing with the algebra $\mathscr{U}(so(2q, \mathbb{C}))$. However, it is easy to see that the commuting operators we have obtained also lie in the universal enveloping algebra $\mathscr{U}$. Since any commuting operator in $\mathscr{U}$ also belongs to $\mathscr{U}(so(2q, \mathbb{C}))$, we have found all generators of the commuting operators in $\mathscr{U}$. Therefore, the proof of the theorem is now complete.

Now, let R^* denote the matrix

$$\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix} R^{\mathrm{T}} \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}.$$

Then we have the following proposition.

Proposition 3.5. The differential operators of the form $\mathrm{tr}(A_1 \ldots A_i \ldots A_r)$, where $A_i = R$ or R^*, $1 \leqslant i \leqslant r$, r is an integer $\geqslant 0$, generate the same algebra $\mathscr{V}$ of commuting differential operators as the differential operators defined by (3.4) in theorem 3.3.

Proof. The proof of this proposition is similar to the proof of lemma 6 in [6].

Remark 3.6. It is not difficult to show that the differential operators of the form $\mathrm{tr}(A_1 \ldots A_i \ldots A_r)$, where $A_i = R$ or R^*, commute with the right action of the subgroup H of G. So, theorem 3.3 and proposition 3.5 illustrate a dual-pair action on $\mathscr{F}$.

To be useful for our programme of multiplicity breaking, the commuting operators defined in theorem 3.3 must be Hermitian. Therefore, we want to know what the adjoints of those operators look like. Instead of examining the adjoints of the operators defined in theorem 3.3 directly, it is more convenient to check the adjoints of the commuting operators using the forms defined in proposition 3.5.

Proposition 3.7. The adjoint of a differential operator $\mathrm{tr}(A_1 \ldots A_r)$ defined in proposition 3.5 is given by $\mathrm{tr}(A_r \ldots A_1)$.

Proof. It was shown in [3] that the adjoint of

$$R_{rs} = \sum_{\eta=1}^{q} Z_{\eta r} \frac{\partial}{\partial Z_{\eta s}} \qquad 1 \leqslant r \qquad s \leqslant 2k$$

is equal to

$$R_{sr} = \sum_{\eta=1}^{q} Z_{\eta s} \frac{\partial}{\partial Z_{\eta r}}.$$

To avoid cumbersome notation, we use the Einstein convention. If we let $R^*_{\alpha\beta}$ denote the $\alpha\beta$ entry of the matrix R^*, then we have

$$\mathrm{Tr}(A_1 \ldots A_r) = \tilde{R}_{\alpha_1\alpha_2} \tilde{R}_{\alpha_2\alpha_3} \ldots \tilde{R}_{\alpha_r\alpha_1}$$

where

$$\tilde{R}_{\alpha_j\alpha_{j+1}} = R_{\alpha_j\alpha_{j+1}} \qquad \text{if } A_j = R$$

or

$$\tilde{R}_{\alpha_j\alpha_{j+1}} = R^*_{\alpha_j\alpha_{j+1}} \qquad \text{if } A_j = R^*.$$

Now, if $\mathcal{O}$ is an operator, then let $\mathcal{O}^+$ denote the adjoint of $\mathcal{O}$. It follows that

$$\begin{aligned}
(\mathrm{Tr}(A_1 \ldots A_r))^+ &= (\tilde{R}_{\alpha_1\alpha_2} \tilde{R}_{\alpha_2\alpha_3} \ldots \tilde{R}_{\alpha_r\alpha_1})^+ \\
&= (\tilde{R}_{\alpha_r\alpha_1})^+ (\tilde{R}_{\alpha_1\alpha_2} \tilde{R}_{\alpha_2\alpha_3} \ldots \tilde{R}_{\alpha_{r-1}\alpha_r})^+ \\
&= \tilde{R}_{\alpha_1\alpha_r} \tilde{R}_{\alpha_r\alpha_{r-1}} \ldots \tilde{R}_{\alpha_{r-1}\alpha_1} \\
&= \mathrm{Tr}(A_r \ldots A_1).
\end{aligned}$$

Hence, the proof is complete.

Remark 3.8. From the above proposition, the adjoint of an operator in $\mathscr{V}$ is still in $\mathscr{V}$. Now, it is obvious that if an operator is not Hermitian, then the sum of the operator and its adjoint would be Hermitian. Therefore, it is always possible to find a Hermitian differential operator in $\mathscr{U}$ that commutes with L_{ij}. Now, all we need to do is to pick a Hermitian operator in $\mathscr{V}$ and use it to decompose the $\mathrm{Ker}^{(m')(m)}_{\max}$ space into distinct one-dimensional subspaces. In general, we choose the commuting differential operator in an *ad hoc* manner. However, in practice, just a low-degree differential operator generally suffices. Moreover, it has been shown in [10] that all the differential operators defined in proposition 3.5 can be generated from a finite set. In the next section, we shall illustrate the procedure by an example.

4. An example

In this section, we are going to consider the multiplicity breaking of the irreducible representation (1, 1, 0, 0) of $Sp(8, \mathbb{C})$ in the irreducible representation (2, 2, 1, 1, 0, 0, 0, 0) of $GL(8, \mathbb{C})$ when we restrict this representation to the subgroup $Sp(8, \mathbb{C})$.

According to a result in [5], the irreducible representation of (1, 1, 0, 0) of $Sp(8, \mathbb{C})$ occurs in this restriction twice. We shall show that our procedure will also allow us to rederive this multiplicity.

According to our programme, we consider the Fock space $\mathscr{F}(\mathbb{C}^{4\times 8})$ which contains the $Sp(8, \mathbb{C})$ module $P^{(2,2,1,1)}(\mathbb{C}^{4\times 8})$. The dual pair for this example is therefore $(SO^*(8), Sp(8, \mathbb{C}))$.

The submodule $P^{(2,2,1,1)}(\mathbb{C}^{4\times 8})$ contains in turn the subspace $V_{GL}^{(2,2,1,1,0,\ldots,0)}$, which consists of polynomial functions in $P^{(2,2,1,1)}(\mathbb{C}^{4\times 8})$ that are simultaneously annihilated by the lowering operators

$$L_{ij}=\sum_{\eta=1}^{8} Z_{i\eta}\frac{\partial}{\partial Z_{j\eta}} \qquad 1\leqslant i<j\leqslant 4.$$

Now, we need to know how many times the $V_{Sp}^{(1,1,0,0)}$ occurs in $P^{(2,2,1,1)}(\mathbb{C}^{4\times 8})$. We are going to prove a general theorem for this purpose. Let n be an arbitrary positive integer. Then, it is well known that the representation $(R_{GL}^{(n,0,\ldots,0)}, V_{GL}^{(n,0,\ldots,0)})$ is irreducible when we restrict it to the subgroup $Sp(2k, \mathbb{C})$. Therefore, $V_{Sp}^{(n,0,\ldots,0)}$ is isomorphic to $V_{GL}^{(n,0,\ldots,0)}$ as the $Sp(2k, \mathbb{C})$ module and we have the following theorem.

Theorem 4.1. Suppose $(m)=(m_1, \ldots, m_q)$ is a q-tuple of non-negative integers such that $m_1\geqslant\ldots\geqslant m_q$. Recall that $P^{(m)}$ is the space of polynomial functions which transform covariantly with respect to $\zeta^{(m)}$. Let $R_0^{(m)}$ denote the representation of $Sp(2k, \mathbb{C})$ on $P^{(m)}$ by right translation. Then the $Sp(2k, \mathbb{C})$ module $P^{(m)}$ is isomorphic to the tensor product $V_{Sp}^{(m_1,0,\ldots,0)}\otimes V_{Sp}^{(m_2,0,\ldots,0)}\otimes\ldots\otimes V_{Sp}^{(m_q,0,\ldots,0)}$.

Proof. It follows easily by the fact that $V_{Sp}^{(m_i,0,\ldots,0)}$ is isomorphic to $V_{GL}^{(m_i,0,\ldots,0)}$ as the $Sp(2k, \mathbb{C})$ module and the polynomial space $P^{(m)}$ is isomorphic to the tensor product $V_{GL}^{(m_1,0,\ldots,0)}\otimes V_{GL}^{(m_2,0,\ldots,0)}\otimes\ldots\otimes V_{GL}^{(m_q,0,\ldots,0)}$.

Remark 4.2. From the above theorem, all we need to know now is the direct sum decomposition of the tensor product $(m_1, 0, \ldots, 0)\otimes\ldots\otimes(m_q, 0, \ldots, 0)$ of $Sp(2k, \mathbb{C})$.

Using a result in [7], we first derived the following formula for the decomposition of the tensor product $(m_1, \ldots, m_k)\otimes(n, 0, \ldots, 0)$, which we call the Weyl formula of $Sp(2k, \mathbb{C})$:

$$(m_1, \ldots, m_k)\otimes(n, 0, \ldots, 0)\cong\sum$$
$$\oplus\ (m_1+a_1-a_{2k}, m_2+a_2-a_{2k-1}, m_3+a_3-a_{2k-2}, \ldots, m_k+a_k-a_{k+1})$$

where the sum is over all integers a_i, $i=1, \ldots, 2k$, subject to the conditions

$$a_1+\ldots+a_{2k}=n$$
$$0\leqslant a_i\leqslant m_{i-1}-m_i-a_{2k-(i-2)}+a_{2k-(i-1)}$$
$$0\leqslant a_{2k-j}\leqslant m_{j+1}-m_{j+2}$$
$$0\leqslant a_{k+1}\leqslant m_k$$

where $i=2, 3, \ldots, k$ and $j=0, 1, \ldots, k-2$.

If $m_1, \ldots, m_k$ and n are big integers, then the tensor product decomposition of $(m_1, \ldots, m_k) \otimes (n, 0, \ldots, 0)$ will involve many terms, and the calculation of such a decomposition is a tedious process. The advantage of the above formula is that we can easily write a computer program to perform the calculation. Now, if we use the above formula repeatedly q times, then we can easily obtain the direct sum decomposition of the tensor product $V_{Sp}^{(m_1,0,\ldots,0)} \otimes V_{Sp}^{(m_2,0,\ldots,0)} \otimes \ldots \otimes V_{Sp}^{(m_q,0,\ldots,0)}$.

Now, according to theorem 4.1 and remark 4.2, $V_{Sp}^{(1,1,0,0)}$ occurs in $P^{(2,2,1,1)}(\mathbb{C}^{4\times 8}) \simeq V_{Sp}^{(2,0,0,0)} \otimes V_{Sp}^{(2,0,0,0)} \otimes V_{Sp}^{(1,0,0,0)} \otimes V_{Sp}^{(1,0,0,0)}$ eight times. Let $h_{\max}$ be the highest-weight vector of $V_{Sp}^{(1,1,0,0)}$; then it is given by

$$h_{\max} = z_{11}z_{22} - z_{12}z_{21}.$$

There exist eight linearly independent intertwining operators, for example $L_{43}P_{23}L_{12}P_{23}$, $L_{31}P_{14}L_{24}P_{14}$, $L_{42}P_{12}L_{31}P_{12}$, $L_{24}P_{34}L_{13}P_{34}$, $L_{24}L_{23}P_{13}P_{14}L_{42}L_{31}$, $L_{41}P_{13}L_{23}P_{13}$, $L_{24}L_{21}L_{32}P_{14}P_{14}$ and $P_{13}L_{13}L_{41}P_{12}L_{31}$, that send the $Sp(8, \mathbb{C})$ module $V_{Sp}^{(1,1,0,0)}$ into the $Sp(8, \mathbb{C})$ module $P^{(2,2,1,1)}(\mathbb{C}^{4\times 8})$.

At this point, we want to mention how to choose the above eight linearly independent operators. Our goal is to find elements in $\mathscr{U}(SO^*(8))$ that send $V_{Sp}^{(1,1,0,0)}$ into $P^{(2,2,1,1)}(\mathbb{C}^{4\times 8})$. In $\mathscr{U}(SO^*(8))$, the raising operators are $P_{\alpha\beta}$ and $L_{\alpha\beta}$ for $\alpha > \beta$ and the lowering operators are $P_{\alpha\beta}$ and $D_{\alpha\beta}$ for $\alpha < \beta$. Therefore, we want to combine certain raising and lowering operators in $\mathscr{U}(SO^*(8))$ so that we can raise the 4-tuple of integers $(1, 1, 0, 0)$ to $(2, 2, 1, 1)$.

Let us return to our example. We use the Casimir operator

$$C = \mathrm{Tr}(RR^*R)$$

where

$$R_{rs} = \sum_{\eta=1}^{4} z_{\eta r} \frac{\partial}{\partial Z_{\eta s}} \qquad 1 \leqslant r, s \leqslant 8.$$

According to proposition 3.7, C is a Hermitian operator and the space $W_{\max} \equiv W_{\max}^{(1,1,0,0)(2,2,1,1,0,0,0,0)}$ is spanned by

$$f_1 = L_{43}P_{23}L_{12}P_{23}h_{\max} \qquad f_2 = L_{31}P_{14}L_{24}P_{14}h_{\max}$$

$$f_3 = L_{42}P_{12}L_{31}P_{12}h_{\max} \qquad f_4 = L_{24}P_{34}L_{13}P_{34}h_{\max}$$

$$f_5 = L_{24}L_{23}P_{13}P_{14}L_{42}L_{31}h_{\max} \qquad f_6 = L_{41}P_{13}L_{23}P_{13}h_{\max}$$

$$f_7 = L_{24}L_{21}L_{32}P_{14}P_{14}h_{\max} \qquad f_8 = P_{13}L_{13}L_{41}P_{12}L_{31}h_{\max}$$

The operators L_{ij}, $1 \leqslant i < j \leqslant 4$, then project $W_{\max}$ on to $\mathrm{Ker}_{\max} \equiv \mathrm{Ker}_{\max}^{(1,1,0,0)(2,2,1,1,0,0,0,0)}$. The application of the operators L_{ij}, $1 \leqslant i < j \leqslant 4$, to a general vector in $W_{\max}$ of the form $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 + \alpha_5 f_5 + \alpha_6 f_6 + \alpha_7 f_7 + \alpha_8 f_8$, $\alpha_i \in \mathbb{C}$, leads to a system of linear equations which in turn implies that $\mathrm{Ker}_{\max}$ has dimension two. The Casimir operator C acting on $\mathrm{Ker}_{\max}$ has two distinct eigenvalues $\lambda_1 = 37/3$ and $\lambda_2 = -21$. The corresponding eigenvector for λ_1 is $h_1 = 300f_1 - 200f_2 - 100f_3 + 683f_4 + 200f_5 + 300f_6 - 50f_7 + 49f_8$ and the corresponding eigenvector for λ_2 is $h_2 = -14f_1 - 7f_4 - 21f_8$. Clearly, h_1 and h_2 are orthogonal vectors since $\lambda_1 \neq \lambda_2$ and C is Hermitian.

In conclusion, the two intertwining maps that send $V_{Sp}^{(1,1,0,0)}$ into two orthogonal (equivalent) submodules of $V_{GL}^{(2,2,1,1,0,\ldots,0)}$ which serve as the labels are

$$P_1 = 300L_{43}P_{23}L_{12}P_{23} - 200L_{31}P_{14}L_{24}P_{14} - 100L_{42}P_{12}L_{31}P_{12} + 683L_{24}P_{34}L_{13}P_{34} + 200L_{24}L_{23}P_{13}P_{14}L_{42}L_{31} + 300L_{41}P_{13}L_{23}P_{13} - 50L_{24}L_{21}L_{32}P_{14}P_{14} + 49P_{13}L_{13}L_{41}P_{12}L_{31}$$

and

$$P_2 = -14L_{43}P_{23}L_{12}P_{23} - 7L_{24}P_{34}L_{13}P_{34} - 21P_{13}L_{13}L_{41}P_{12}L_{31}$$

which are obtained from the forms of h_1 and h_2 in terms of L_{ij}, P_{ij}.

5. Final remarks

This paper is different from [3, 5, 11, 14] in the sense that we did not give another way of calculating the multiplicity occurring in the branching rule $GL(2k, \mathbb{C})\downarrow Sp(2k, \mathbb{C})$. Instead, we assume the multiplicity is known and we want to distinguish the equivalent representations that occur in the branching rule. We have shown how to break the multiplicity that occurs in the branching rule, by finding generalized Casimir operators whose eigenvalues and eigenvectors can be used as labels to resolve the ambiguity occurring when equivalent representations appear more than once in the branching rule. The procedure given in section 3 is possible to implement on a computer, and our immediate goal is to write a computer program for the above procedure. In fact, some of the calculations in section 4 were computed using the computer program Mathematica. Though we have restricted our attention in this paper to the branching rule $GL(2k, \mathbb{C})\downarrow Sp(2k, \mathbb{C})$, our procedure can be used on other branching rules with certain modifications. We intend to investigate these problems in future publications.

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